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METHOD OF EXACT LINEARIZATION OF NONLINEAR AUTONOMOUS DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Necessary and sufficient conditions are established for exact linearization of nonlinear autonomous second order differential equations, using the nonlinear transformation of the function and the independent variable. The problem of the calculus of variation of determining trajectories of a point in a conservative field, the Lotki—Volterra system which defines dynamics of two interacting biological populations, dynamic systems with separated variables, and dynamic systems of the Liouville type are considered as examples of proposed method applications.

1. Statement of the problem and general results. Let us consider the autonomous nonlinear second order differential equation

$$N(x) \equiv x^{\cdot\cdot} + f(x)x^{\cdot 2} + \varphi(x)x^{\cdot} + \psi(x) = 0 \tag{1.1}$$

and construct the class of equations of the type (1.1) which depends on two arbitrary functions, and whose solutions are expressed in terms of quadratures. The construction is realized on the premise of the following meaning of exact linearization. Using the transformation of dependent and independent variables

$$x \to X = v^{-1}(x)x, \quad dt \to d\tau = u(x)dt \tag{1.2}$$

$$u(x(t))v(x(t)) \neq 0, \quad \forall t \in I = \{t \mid a \leqslant t \leqslant b\}$$

the sought class of equations of the type (1.1) is reduced to the preassigned linear autonomous form

$$X'' + b_1X' + b_0X + c = 0, \quad b_1, b_0, c = \text{const}, \quad (') = d / d\tau \tag{1.3}$$

As the result of linearization, the investigation of the nonlinear equation of the type (1.1) in the plane of variables (x, t) reduces to the analysis of the linear equation (1.3) in the plane (X, τ), and the application of inverse transformation of (1.2).

Linearization by transformation of the unknown function was used in [1], and by transformation of the independent variable was applied in [2—4]. Individual examples of [equations of] type (1.1) were considered in [5—8].

T h e o r e m 1. For reducing Eq. (1.1) to the form (1.3) by transform (1.2) it is necessary and sufficient that (1.1) can be factorized in terms of first order operators of the form

$$\left(D - \frac{v^{\cdot}}{v} - r_2u - \frac{u^{\cdot}}{u}\right)\left(D - \frac{v^{\cdot}}{v} - r_1u\right)x + cu^2v = 0, \quad D = \frac{d}{dt} \tag{1.4}$$

(here the operators are generally noncommutative) or that (1.1) could be factorized in the form

$$(u^{-1}D - u^{-1}v^{-1}v^{\cdot} - r_2)(u^{-1}D - u^{-1}v^{-1}v^{\cdot} - r_1)x + cv = 0 \qquad (1.5)$$

using commutative operators. In this equation $v^{\cdot} = v^{*}x^{\cdot}$, $u^{\cdot} = u^{*}x^{\cdot}$, and $(*) = d / dx$, r_k are roots of the characteristic equation

$$r^2 + b_1 r + b_0 = 0 \qquad (1.6)$$

The proof of Theorem 1 is similar to that in [9,10] of the corresponding theorem for linear nonautonomous differential equations using the method of differential operator factorization.

The necessity. Let the application of (1.2) to (1.1) yield formula (1.3) which we write in the form

$$(D_\tau - r_2)(D_\tau - r_1)X + c = 0, \quad D_\tau = d / d\tau \qquad (1.7)$$

Multiplying the left-hand side of (1.7) by u^2v, using transform (1.2), and applying to the obtained expression the operator identity

$$(u^{-1}D - r_k)\, u^{1-k}v^{-1} = u^{-k}v^{-1}L_k$$
$$L_k = D - \frac{v^{\cdot}}{v} - r_k u - (k-1)\frac{u^{\cdot}}{u}, \quad k = 1, 2$$

we obtain (1.4).

The sufficiency. Let us apply (1.2) to (1.4) and use the identity

$$L_k u^{k-1} v = u^k v\,(D_\tau - r_k), \quad k = 1, 2$$

This brings (1.4) to the form

$$u^2 v\,(D_\tau - r_2)(D_\tau - r_1)X + cu^2 v = 0$$

from which follows (1.3). To pass from expansion (1.4) to (1.5) it is necessary to apply successively the operator identity

$$u^{-k}L_k = (u^{-1}D - (vu)^{-1}v^{\cdot} - r_k)u^{-k-1}$$

The commutativity of operators $u^{-1}D - (vu)^{-1}v^{\cdot} - r_k$ can be tested directly. To prove the theorem it remains only to point out that condition (1.5), as well as (1.4), is not only necessary but also sufficient.

The above theorem indicates the analogy between the linearizable nonlinear autonomous equations of type (1.1) and the algebraic equations (1.6). Previously [9,10] the method of factorization resulted in the establishment of analogy also for linear nonautonomous differential equations that are reducible to equations with constant coefficients.

Lemma 1. If (1.1) can be linearized by (1.2), the following expansion holds;

$$EN(x) \equiv Ex^{\cdot\cdot} - Fx^{\cdot 2} + Gx^{\cdot} + H \qquad (1.8)$$
$$v \neq ax, \quad E = 1 - v^{-1}v^{*}x, \quad F = (2v^{-1}v^{*} + u^{-1}u^{*})E + xv^{-1}v^{**}$$
$$G = b_1 uE, \quad H = b_0 u^2 x + cvu^2$$

To prove this it is sufficient to multiply out the differential operators in formuls (1.4)

L e m m a 2. The general solution of the nonlinear autonomous second order equation

$$v^{**} - 2v^{-1}v^{*2} + (2x^{-1} - u^{-1}u^{*} - f)v^{*} + x^{-1}(u^{-1}u^{*} + f)v = 0 \qquad (1.9)$$
$$f = f(x), \quad v \neq ax + b$$

is of the form (α and β are arbitrary constants)

$$v(x) = x\left[\alpha + \beta \int u \exp\left(\int f dx\right) dx\right]^{-1} \qquad (1.10)$$

P r o o f. The substitution $v = V^{-1}$ reduces Eq. (1.9) to the linear nonautonomous form

$$V^{**} + \left(\frac{2}{x} - \frac{u^*}{u} - f\right)V^* - \frac{1}{x}\left(\frac{u^*}{u} + f\right)V = 0 \qquad (1.11)$$

which admits the factorization

$$\left(D_x + \frac{1}{x} - \frac{u^*}{u} - f\right)\left(D_x + \frac{1}{x}\right)V = 0, \; D_x = \frac{1}{x}$$

Hence the general solution of (1.11) is of the form

$$V = \frac{1}{x}\left[\alpha + \beta \int u \exp\left(\int f dx\right) dx\right]$$

where α and β are arbitrary constants, and consequently $v(x)$ satisfies the relation (1.10).

T h e o r e m 2. For Eq. (1.1) to be linearized by transformation (1.2) it is necessary and sufficient that it is representable in one of the following forms:

$$x^{\cdot\cdot} + fx^{\cdot 2} + b_1\varphi x^{\cdot} + \Lambda_1(\varphi, x) = 0 \qquad (1.12)$$

$$\Lambda_1(\varphi, x) = \varphi \exp\left(-\int f dx\right)\left[b_0 \int \varphi \exp\left(\int f dx\right) dx + \frac{c}{\beta}\right]$$

$$x^{\cdot\cdot} - \left(\frac{2a}{ax+b} + \frac{\varphi^*}{\varphi}\right)x^{\cdot 2} + b_1\varphi x^{\cdot} + \Lambda_2(\varphi, x) = 0 \qquad (1.13)$$

$$\Lambda_2(\varphi, x) = \varphi^2 (ax + b)b^{-1}[b_0 x + c(ax + b)]$$

Equations (1.12) and (1.13) are reduced to (1.3), respectively, by the transform

$$X_1(x) = \beta \int \varphi \exp\left(\int f dx\right) dx, \quad d\tau = \varphi(x) dt$$
$$X_2(x) = x / (ax + b), \quad b \neq 0, \quad d\tau = \varphi(x) dt$$

P r o o f. We write (1.9) in the form

$$[(2v^{-1}v^{*} + u^{-1}u^{*})(1 - v^{-1}v^{*}x) + xv^{-1}v^{**}] / (1 - v^{-1}v^{*}x) = f$$

Setting in formula (1.10) $\alpha = 0$ and $\varphi \equiv u(x)$ and substituting the expression for X_1 in (1.8), we obtain (1.2). Equation (1.13) is obtained by the substitution into (1.8) of the expression

$$v = ax + b, \quad b \neq 0 \tag{1.14}$$

Corollary 1. The general solution of Eqs. (1.12) and (1.13) can be represented in the parametric form

$$\begin{aligned} &r_1 \neq r_2 \neq 0, \quad X_i = C_1 \exp(r_1\tau) + C_2 \exp(r_2\tau) - c/b_0 \\ &r_1 = r_2 = -b_1/2 \neq 0, \quad X_i = \exp(-b_1\tau/2)(C_1\tau + C_2) - c/b_0 \\ &r_1 = 0, \quad r_2 \neq 0, \quad X_i = C_1 + C_2 \exp(-b_1\tau) - cb_1^{-1}\tau \\ &r_1 = r_2 = 0, \quad X_i = C_1 + C_2\tau - \frac{c}{2}\tau^2 \\ &b_1 = 0, \quad b_0 > 0, \quad X_i = A \sin(\sqrt{b_0}\,\tau + B) - c/b_0 \\ &b_1 = 0, \quad b_0 < 0, \quad X_i = A \operatorname{sh}(\sqrt{-b_0}\,\tau + B) - c/b_0 \\ &i = 1, 2; \quad t = \int \frac{d\tau}{\varphi(x(\tau))} \end{aligned} \tag{1.15}$$

where C_1, C_2, A, and B are arbitrary constants.

Note that the general solution of Eqs. (1.12) and (1.13) obtained by the elimination of parameter τ from Eqs. (1.15) are nonlinear functions of two arbitrary constants.

Corollary 2. If $c = 0$, Eqs. (1.12) and (1.13) have one-parameter solutions $(k = 1, 2)$

$$r_k t + C_k^{(1)} = I_1, \quad I_1 = \int \frac{\exp \lambda(x)\, dx}{\beta^{-1} X_1(x)}, \quad \lambda(x) = \int f\, dx \tag{1.16}$$

$$r_k t + C_k^{(2)} = I_2, \quad I_2 = b \int \frac{dx}{x(ax+b)\varphi} \tag{1.17}$$

where $C_k^{(i)}$ is an arbitrary constant which for $i = 1, 2$ satisfies the respective first order equations

$$x^{\cdot} - r_k \exp\left(-\int f\, dx\right) \int \varphi \exp\left(\int f\, dx\right) dx = 0 \tag{1.18}$$

$$x^{\cdot} - r_k b^{-1} x (ax + b) \varphi = 0 \tag{1.19}$$

where r_k are simple characteristic roots of (1.6), and when (1.6) has multiple roots, they are solutions of the form

$$-\frac{b_1}{2} t + (k-1) \int \frac{dt}{\tau(t)} + C_k^{(i)} = I_i, \quad i = 1, 2 \tag{1.20}$$

where $\tau(t)$ is the inversion of the integral for t in (1.15).

Proof. Equations (1.12) and (1.13) are equivalent to factorization of (1.15) with $c = 0$ to which for $r_1 \neq r_2$ corresponds the system of first order equations

$$(1 - v^{-1}v^* x)\, x^{\cdot} - r_k u x = 0, \quad k = 1, 2$$

From this, when $u \equiv \varphi$, on the strength of (1.10) or (1.14) we obtain, respectively, Eqs. (1.18) and (1.19) Let now $r_1 = r_2 = -b_1/2$. Then the relations

$$\exp(-b_1/2)\,\tau^{k-1} = X_i(x), \quad \tau = \int \varphi(x)\, dt, \quad k = 1, 2 \tag{1.21}$$

are valid, and from them follow formulas (1.20).

To illustrate this we consider two examples.

Example 1. Let us consider the equation

$$x^{\cdot\cdot} + 3xx^{\cdot} + x^3 = 0 \tag{1.22}$$

which occurs in investigations of univalued functions defined by second order differential equations [11]. This equation belongs to the class (1.12), where

$$f(x) = 0,\ \varphi(x) = x,\ b_1 = 3,\ c = 0,\ \beta = 2,\ \text{and}\ b_0 = 2$$

and admits the factorization

$$(D - r_2 x)(D + x^{\cdot} / x - r_1 x)\, x \equiv 2\,(x^{\cdot\cdot} + 3xx^{\cdot} + x^3)$$

obtained from (1.8) for $v = x^{-1}$ and $u = x$ (r_1 and r_2 are roots of the characteristic equation $r^2 + 3r + 2 = 0$). Let $r_1 = -2$ and $r_2 = -1$. Then the one-parameter solutions of Eq. (1.22) by virtue of (1.16) are of the form

$$x = 1 / (t + c), \quad x = 2 / (t + c) \tag{1.23}$$

Note that solution (1.23) does not appear in [11].

The substitution $x^2 = X$, $d\tau = x dt$ reduces Eq. (1.22) to the linear form $X'' + 3X' + 2X = 0$ from which on the strength of (1.15) we have

$$x = \pm [C_1 \exp(-2\tau) + C_2 \exp(-\tau)]^{1/2}, \quad t = \int \frac{d\tau}{x} \tag{1.24}$$

where C_1 and C_2 are arbitrary constants.

If $C_2 \neq 0$

$$x = \pm \exp(-\tau)(C_1 + C_2 \exp \tau)^{1/2}, \quad t = \pm 2 C_2^{-1} (C_1 + C_2 \exp \tau)^{1/2} - k$$

where k is an arbitrary constant. Eliminating parameter τ and using the notation $2k = b$ and $k^2 - 4C_1 / C_2^2 = c$, we obtain a two-parameter set of solutions $x = (2t + b) / (t^2 + bt + C)$. If $C_2 = 0$ (1.24) yields formula (1.23).

Example 2. We consider one-dimensional motion of a particle along the axis. Let M be the mass of the central body and m the mass of a particle. The equation of one-dimensional unperturbed motion of the particle and the energy integral are of the form

$$x^{\cdot\cdot} + x^{\cdot 2} / 2x + h_k / x = 0, \quad h_k = k^2 / x - x^{\cdot 2} / 2 \tag{1.25}$$

where h_k is the Keplerian energy $k^2 = G(M + m)$ and G the gravitational constant. Equation (1.25) has a singularity at $x = 0$. It is reduced by the transformation $x = \sqrt{x} y$, $d\tau = x^{-1} dt$ to the form of the linear harmonic oscillator $y'' + h_k y / 2 = 0$. Thus the method of exact linearization makes possible the regularization of differential equations. A more cumbersome method was used in [12] for linearization.

Corollary 3. General solutions of Eqs. (1.12) and (1.13) for ($b_1 = 0$) are represented, respectively, by the following two relations between t and x:

$$t = \pm \frac{\beta}{\sqrt{|b_0|}} \int \frac{\exp[\lambda(x)]\, dx}{[A_1 \mp (c/b_0 + X_1(x))^2]^{1/2}} + A_2 \tag{1.26}$$

$$t = \pm \frac{b_0}{\sqrt{|b_0|}} \int \frac{dx}{\varphi(ax + b)^2 [A_1 \mp (c / b_0 + X_2(x))^2]^{1/2}} + A_2$$

where A_1 and A_2 are arbitrary constants and the signs plus and minus in the radicands correspond, respectively, to $b_0 > 0$ and $b_0 < 0$.

P r o o f. Solutions (1.26) of Eqs. (1.12) and (1.13) are obtained by eliminating parameter τ from the last three of formulas (1.15).

T h e o r e m 3. 1) The kernel $u(x(t))$ and multiplier $v(x(t))$ of transform (1.2) which linearizes Eq. (1.12) satisfy the respective equations

$$u^{\cdot\cdot} + E(\Phi) u^{\cdot 2} + b_1 u u^{\cdot} + \Phi^{*-1}\Delta_1(u, \Phi) = 0$$
$$v^{\cdot\cdot} + E(F) v^{\cdot 2} + b_1\varphi(F) v^{\cdot} + F^{*-1}\Lambda_1(\varphi(F), F) = 0$$
$$E(y) = y^{*-1}y^{**} + f(y)y^*$$

where in the first equation $(*) = d/du$, $x = \Phi(u)$ is a function inverse of $u = \varphi(x)$, and in the second $(*) = d/dv$ and $x = F(v)$ represents the inversion of the expression

$$v(x) = x\left[\beta\int\varphi\exp\left(\int f\,dx\right)dx\right]^{-1}$$

2) The kernel $u(x(t))$ and multiplier $v(x(t))$ of transform (1.2) which linearizes Eq. (1.13) satisfy the respective equations

$$u^{\cdot\cdot} + \left(\Phi^{*-1}\Phi^{**} - \frac{2a\Phi^*}{a\Phi + b} - \frac{1}{u}\right)u^{\cdot 2} + b_1 u u^{\cdot} + \Phi^{*-1}\Lambda_2(u, \Phi) = 0$$
$$v^{\cdot\cdot} - \left(\frac{2}{v} + \frac{\varphi^*}{\varphi}\right)v^{\cdot 2} + b_1\varphi v^{\cdot} + a\Lambda_2\left(\varphi, \frac{(v-b)}{a}\right) = 0$$

P r o o f. Part 1 is proved by substituting $x = \Phi(u)$ and $x = F(v)$ into Eq. (1.12), and 2) is proved by substituting $x = \Phi(u)$ and $x = (v - b)/a$ into Eq. (1.13).

The following theorem whose proof is obtained by direct substitution shows that the term containing the square of derivative can always be eliminated from Eqs. (1.12) and (1.13).

T h e o r e m 4. The transforms

$$x = s(y), \quad s^* = \exp\left(-\int f(s)\,ds\right)$$
$$s^* = (as + b)^2\varphi(s), \quad (*) = d/dy$$

reduce Eqs. (1.12) and (1.13) to respective equations of the Liénard type.

$$y^{\cdot\cdot} + b_1\varphi(s)y^{\cdot} + \varphi(s)\left(\frac{c}{\beta} + b_0\int\varphi(s)\,dy\right) = 0$$
$$y^{\cdot\cdot} + b_1\varphi(s)y^{\cdot} + \varphi(s)\left(\frac{b_0 s}{b(as+b)} + \frac{c}{\beta}\right) = 0$$

2. Linearization of certain classes of dynamic systems. Let us consider the following classes.

1°. The class of equations

$$x^{\cdot\cdot} - \varphi^*\varphi^{-1}x^{\cdot 2} + b_1\varphi x^{\cdot} + \varphi^2(b_0 x + c/\beta) = 0$$

that can be linearized by transformation of the independent variable. The respective transform is of the form $X = \beta x,\ d\tau = \varphi dt$.

2°. The classes of dynamic systems of the type (1.1) equivalent to (1.12) in which functions $f(x)$ and $\psi(x)$, or $\varphi(x)$ and $\psi(x)$ are taken, respectively, as the independent variable.

Let $f(x)$ and $\psi(x)$ be arbitrary functions. The corresponding class of dynamic systems is defined by the equation

$$x^{\cdot\cdot} + fx^{\cdot 2} + b_1\Omega + b_0\psi = 0, \quad b_0 \neq 0 \tag{2.1}$$

$$\Omega = \psi \exp(\lambda(x))[2\textstyle\int\psi \exp(2\lambda(x))\, dx]^{-1/2}$$

linearized by the transform

$$X = [2\textstyle\int\psi \exp(2\lambda(x))\, dx]^{1/2}, \quad d\tau = \Omega dt \tag{2.2}$$

where $\lambda(x)$ is defined in (1.17).

One more class of dynamic systems with arbitrary $f(x)$ and $\psi(x)$ is defined by the equation

$$x^{\cdot\cdot} + fx^{\cdot 2} + b_1\psi \exp(\textstyle\int f dx)\, x^{\cdot} + c\beta^{-1}\psi = 0 \tag{2.3}$$

which can be linearized by the transform

$$X = \beta \int \psi \exp\left(\int f\, dx\right) dx, \quad d\tau = \psi \exp\left(\int f\, dx\right) dt$$

Finally, the class of dynamic systems with arbitrary $\varphi(x)$ and $\psi(x)$ is defined by the equation

$$x^{\cdot\cdot} + \left(b_0 \frac{\varphi^2}{\psi} - \frac{\psi^*}{\psi} + \frac{\varphi^*}{\varphi}\right) x^{\cdot 2} + b_1\varphi x^{\cdot} + \psi = 0$$

which can be linearized by the transform

$$X = \exp\left(b_0 \int \frac{\varphi^2}{\psi}\, dx\right), \quad d\tau = \varphi\, dt$$

3°. An important class of dynamic systems is defined by the equation of the form

$$x^{\cdot\cdot} + fx^{\cdot 2} \pm a^2\psi = 0 \tag{2.4}$$

which be transform (2.2) is reduced to the linear equation

$$X'' \pm a^2 X = 0 \tag{2.5}$$

By corollaries 3 and 1 (see Sect. 1) Eq. (2.4) has the first integrals

$$x^{\cdot 2} = a^2\left(C \mp 2 \int \psi \exp\left(2 \int f\, dx\right) dx\right) \exp\left(-2 \int f\, dx\right) \tag{2.6}$$

and in addition admits the one-parameter solutions

$$\int \frac{\Omega}{\psi}\, dx = \pm \sqrt{\mp a^2}\, t + C \tag{2.7}$$

where C is an arbitrary constant and Ω is defined by formula (2.1).

E x a m p l e 3 [13]. Let us determine the first integrals and one-parameter sets of solutions of equation

$$x^{\cdot\cdot} + \frac{1}{nx(1-x^2)} x^{\cdot 2} + \frac{nx^2-1}{nx} = 0 \tag{2.8}$$

In conformity with (2.6) and (2.7) the first integrals and one-parameter solutions of Eq. (2.8) are of the form

$$x^{\cdot 2} = 1 - x^2 + C(x^{-2}-1)^{1/n}, \quad \int \frac{dx}{\sqrt{-1}\sqrt{1-x^2}} = \pm\sqrt{-1}\, t + C$$

Hence Eq. (2.8) has periodic solutions of the form

$$x = \sin(t+C), \quad x = \cos(t+C)$$

4°. A particular case of Eq. (2.4) is provided by the problem of variational calculus on the determination of trajectories of a point moving in a conservative field, for which Euler's equation reduces to the form

$$y'' + f(y)\, y'^2 + f(y) = 0$$

5°. Linearization is also possible in special cases of dynamic systems of the form

$$x_1^{\cdot} = P(x_1, x_2), \quad x_2^{\cdot} = Q(x_1, x_2)$$

which by the elimination of variables are reduced to equations of the form (1.12) and (1.13) or to Eqs. (2.1)—(2.3) that are equivalent to (1.12). To eliminate variables (separate motions) in nonlinear systems we use the method of resultant of differential polynomials.

For example, the Lotki—Volterra system [14]

$$x_i^{\cdot} = \alpha_i x_i + \beta_i x_i x_{i+1}, \quad \alpha_i,\ \beta_i = \text{const}, \quad i = 1,\ 2, \quad i+1 \equiv 0 \pmod 2$$

which defines the dynamics of two interacting biological populations reduces to the system of second order equations

$$x_i^{\cdot\cdot} - x_i^{-1} x_i^{\cdot 2} - (\alpha_{i+1} + \beta_{i+1} x_i)\, x_i^{\cdot} + \alpha_i x_i (\alpha_{i+1} + \beta_{i+1} x_i) = 0, \quad i = 1,\ 2 \tag{2.9}$$

which belong to class (1.12).

For the derivation of (2.9) we consider the auxilliary system

$$x_{i+1}^{\cdot} - (\alpha_{i+1} + \beta_{i+1} x_i)\, x_{i+1} = 0$$
$$\beta_i x_i x_{i+1} + \alpha_i x_i - x_i^{\cdot} = 0$$
$$\beta_i x_i x^{\cdot}_{i+1} + \beta_i x_i^{\cdot} x_{i+1} + \alpha_i x_i^{\cdot} - x_i^{\cdot\cdot} = 0$$

from which follows formula (2.9) for the resultant.

6°. A particular class of dynamic systems that are solvable in quadratures is represented by systems with separated variables for which the kinetic and potential energies are defined by

$$T = \frac{1}{2}\sum_{i=1}^{n} a_i(q_i)\, q_i^{\cdot 2}, \quad U = -\sum_{i=1}^{n} d_i(q_i)$$

where q_i are generalized coordinates.

The Lagrange equations of such systems reduce to equations with separated variables of the form

$$q_i^{\cdot\cdot} + \frac{1}{2}\frac{a_i^*}{a_i} q_i^{\cdot 2} + \frac{1}{a_i} d_i^* = 0, \quad i = 1, \ldots, n, \quad (*) = \frac{d}{dq_i} \tag{2.10}$$

which by transform

$$Q_i = \sqrt{2d_i}, \quad d\tau = d_i^* (2a_i d_i)^{-1/2} dt$$

is reduced to the system of linear equations $Q_i'' + Q_i = 0$, $(') = d / d\tau$, that are of the form (2.5). Using formula (2.6) we obtain for the first integrals the following dependence of q_i on t:

$$dt = \pm a_i^{1/2} (c_i - 2d_i)^{-1/2} dq_i$$

3. **Linearization of Liouville systems.** Let us consider the Liouville systems [15, 16] for which the kinetic and potential energies are, respectively, of the form

$$T = \frac{1}{2} b(q) \sum_{i=1}^{n} a_i(q_i) \dot{q}_i^2, \quad U = -\sum_{i=1}^{n} \frac{d_i(q_i)}{b(q)}, \quad b(q) = \sum_{i=1}^{n} b_i(q_i)$$

Taking into account the energy integral $T - U = h$ we obtain on the strength of Lagrange equations the system of differential equations

$$\ddot{q}_i + \left(\frac{1}{2} \dot{q}_i a_i^{-1} \frac{da_i}{dq_i} + b^{-1} \frac{db}{dt}\right) \dot{q}_i - b^{-2} a^{-1} \frac{d}{dq_i} (hb_i - d_i) = 0 \qquad (3.1)$$

If the coefficients at $\dot{q}_i$ in Eqs. (3.1) are assumed to be functions of t, it is possible to consider the left-hand of each of Eqs. (3.1) as the sum of two expressions, of which $\ddot{q}_i + ({}^1/_2 \dot{q}_i a_i^{-1} da_i / dq_i + b^{-1} db / dt)$ is linear and autonomous, and $b^{-2} a_i^{-1} d (hb_i - d_i) / dq_i$ nonlinear.

We apply to (3.1) the method proposed in [17] for reduction to autonomous form. Using transform

$$d\tau_i = b^{-1}(q) a_i^{-1/2}(q_i) dt \qquad (3.2)$$

we reduce system (3.1) to the system of equations

$$d^2 q_i / d\tau_i^2 - d (hb_i - d_i) / dq_i = 0 \qquad (3.3)$$

which belongs to class (2.4), and transform

$$Q_i = \sqrt{2(hb_i - d_i)}, \quad d\tau_i = -\frac{d}{dq_i} \sqrt{2(hb_i - d_i)}\, ds$$

reduce it to the linear system

$$Q_i'' - Q_i = 0, \quad (') = d / ds$$

By virtue of (2.6) solutions of (3.3) are defined by formulas

$$d\tau_i = \pm [2 (c_i + hb_i - d_i)]^{-1/2} dq_i \qquad (3.4)$$

From (3.2) and (3.4) we also obtain the known system of first integrals

$$\frac{1}{2} b^2 a_i \dot{q}_i^2 = hb_i - d_i + c_i, \quad i = 1, \ldots, n, \quad \sum_{i=1}^{n} c_i = 0$$

and the known relations

$$\frac{\sqrt{a_1}\, dq_1}{\sqrt{2(c_1 + hb_1 - d_1)}} = \ldots = \frac{\sqrt{a_n}\, dq_n}{\sqrt{2(c_n + hb_n - d_n)}}$$

In concluding we point out that the method of exact linearization can be extended to scalar and vector nonlinear differential equations of the second and higher orders, both autonomous and nonautonomous.

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